

REALISM AND THE INCOMPLETENESS THEOREMS IN KURT GÖDEL'S PHILOSOPHY  
OF MATHEMATICS

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by

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## PRÉCIS

This paper stems from an interest in mathematics, the philosophy of mathematics, and Kurt Gödel, an important figure in both disciplines. While possessing previous knowledge of Gödel's famous incompleteness theorems, I first became aware of his philosophy, incidentally, through attending a seminar on the life and work of Alan Turing. Thus having some acquaintance with Gödel's results in the foundations of mathematics and his philosophical views, I began to inquire into the connection, if any, between the two, with a specific focus on his realism or Platonism, the view that mathematical objects have an objective existence. The present study is the culmination (at least in part) of that inquiry.

Rather than viewing Gödel's realism from outside of mathematics, from a purely philosophical perspective, this paper draws attention toward the role that intra-mathematical considerations, i.e., mathematical results and methodology, play in the development of Gödel's position. Of particular interest is the connection between Gödel's incompleteness theorems and realism. I argue that Gödel's intra-mathematical considerations both motivate and strengthen his argument for realism.

To that end, I first provide an overview of the incompleteness theorems themselves in order to provide a foundation from which to survey their implications for realism. Second, I investigate the relation between Gödel's work (both the incompleteness theorems and other results) and his realism and how they influenced one another. Third, in setting the larger

mathematical and philosophical contexts in which Gödel's theorems fit, I examine the history and development of the positions competing with realism in the philosophy of mathematics, highlighting a number of the problems Gödel's theorems pose for each. Fourth, I consider Gödel's Gibbs lecture, where he develops some of the positive implications of his theorems for realism in the form of a disjunctive conclusion and gives other arguments for realism independent of the incompleteness results but similarly supported by developments in the foundations of mathematics. Finally, I briefly consider some of the issues involved in providing a realist epistemology.

Numerous topics for future research in this area readily present themselves. One such suggestion I hope to pursue includes examining the correspondence between Gödel and Rudolf Carnap, a central figure in the development of Logical Positivism.

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## TABLE OF CONTENTS

Précis.....	3
Acknowledgements.....	5
Introduction.....	7
The Incompleteness Theorems.....	9
Realism, Formalism, and Gödel's Results.....	12
Intuitionism.....	20
Logicism and Conventionalism.....	25
The Gibbs Lecture (Gödel's Disjunctive Conclusion).....	33
Epistemology.....	44
Conclusion.....	51
References.....	53

## INTRODUCTION

At the center of Kurt Gödel's philosophy of mathematics is realism or Platonism, the view that, in Gödel's words, mathematical objects and "concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe" ([1951], p. 356). On this account, mathematical entities are abstract, outside of physical space, but are as real as any physical object. As Gödel notes, "The objects and theorems of mathematics are as objective and independent of our free choice and our creative acts as is the physical world" ([1951], p. 312, n. 17). This position stands, at various points, in fundamental disagreement with the other major schools of thought—formalism, intuitionism, and logicism—and has often suffered scorn.

Bertrand Russell, upon meeting Gödel while visiting Princeton's Institute for Advanced Study, remarked in his autobiography, "Gödel turned out to be an unadulterated Platonist, and apparently believed that an eternal 'not' was laid up in heaven, where virtuous logicians might hope to meet it hereafter" ([1968], p. 356). To this Gödel replied:

Concerning my "unadulterated" Platonism, it is no more unadulterated than Russell's own in 1921 when in the *Introduction to Mathematical Philosophy*... he said, "Logic is concerned with the real world just as truly as zoology, though with its more abstract and general features." At that time evidently Russell had met the "not" even in this world, but later on under the influence of Wittgenstein he chose to overlook it.<sup>1</sup>

Though Russell was quick to dismiss Gödel's views, Gödel's Platonism is connected to his work in mathematics and its foundations, and his accomplishments in these disciplines are universally held in high esteem. In fact, the link between is so close that initial appraisals of Platonism as

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<sup>1</sup> Letter draft to Kenneth Blackwell, 1971, in *CW IV*, p. 317. The quotation is from Russell [1919], p. 169.

scandalous ought at the very least to be reconsidered, as Gödel himself formulated an argument for Platonism on the basis of his celebrated incompleteness theorems.

Gödel's defense stems initially both from rigorous mathematical results and faithfulness to mathematical practice. He goes on to contend for Platonism by drawing from concerns outside of mathematics, e.g., as he develops certain philosophical implications of his incompleteness theorems, and this argument is not immune to criticism. Two lines of reasoning, then, are particularly evident in Gödel's thought, and may be distinguished broadly as intra-mathematical (practical), representing the former, and extra-mathematical (philosophical), representing the latter. This paper will argue that Gödel's intra-mathematical considerations both motivate and strengthen his extra-mathematical argument for realism.

To support this thesis, first presented is an overview of the incompleteness theorems. This will provide a foundation from which to survey the implications of the theorems for realism. Following this is an investigation of the relation between Gödel's work (both the incompleteness theorems and other results) and his realism, setting the context in which Gödel's intra-mathematical and extra-mathematical theorizing may be properly understood. After this, a number of the problems Gödel's results pose for the competing positions of formalism, intuitionism, and logicism are highlighted. Attention next is given to Gödel's Gibbs lecture. There he develops some of the implications of his theorems for realism in the form of a disjunctive conclusion and gives other positive arguments for realism independent of the incompleteness results but supported by developments in the foundations of mathematics nonetheless. Finally, the task of providing a realist epistemology is undertaken.

As suggested in Gödel's statements above, the realism in question is defined as the view that at least some mathematical entities exist objectively, independent of the minds, conventions,

and languages of mathematicians. More specific details of this realism are left undefined. For example, the query about which mathematical objects exist—numbers, functions, Hilbert spaces, the null set?—is not answered. In other words, intramural debates among realists, between, say, Frege<sup>2</sup> and Quine,<sup>3</sup> are not addressed. Similarly, an analysis of Gödel’s “conceptual realism” is withheld, since his ontology of concepts, or properties and relations, is a unique feature of his philosophy and does not affect his defense of realism.<sup>4</sup>

## THE INCOMPLETENESS THEOREMS

Before discussing implications of Gödel’s incompleteness theorems for realism, it is necessary to know what the theorems say.<sup>5</sup> In 1931 Gödel published a paper entitled *Über formal unentscheidbare Sätze der Principia mathematica und verwandter Systeme I* [1931],<sup>6</sup> containing a presentation and proof of his two incompleteness theorems, to be submitted the following year as his *Habilitationsschrift* to the University of Vienna. The incompleteness theorems are mathematical results about formal systems of mathematics. In a formal system, all

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<sup>2</sup> Frege [1884].

<sup>3</sup> Quine [1951a], [1953], and [1960], though his views are spread throughout his writings.

<sup>4</sup> This approach is not inimical to Gödel’s own. As Wang recounts, Gödel wishes to demonstrate, at least initially, that there is a kind of realism that cannot be explained away: “There is a weak kind of Platonism that cannot be denied by anybody.... There are four hundred possibilities: e.g., Platonism for integers only, also for the continuum, also for sets, and also for concepts” ([1996], p. 212). Another statement attributed by Gödel to Bernays reads: “It is just as much an objective fact that the flower has five petals as that its color is red.... The idea is not to determine fully a unique objectivism, but to indicate a weak kind” ([1996], p. 212). Once this is granted, a more robust realist might argue that mathematics in fact requires a more extensive realism. See Wang [1991] for a development of this theme.

<sup>5</sup> The results and implications of the theorems are commonly misunderstood. For an excellent treatment of the various uses and misuses of the theorems, see Franzén [2005].

<sup>6</sup> The title is translated *On formally undecidable propositions of Principia Mathematica and related systems I*. A part II of the paper was planned, but apparently never written.

methods of constructing formulas to express mathematical propositions and all axioms to be used in proving theorems are explicitly governed by clearly stated rules. The formal system of *Principia Mathematica* [1925] (*PM*) was Russell and Whitehead's attempt at a comprehensive reproduction of all of mathematics by purely logical means from logical axioms and rules of inference—the essence of the logicist program.<sup>7</sup> Gödel's results disrupted their efforts by showing that there can be no finitely axiomatizable, consistent, and complete formal system after the manner of *PM*.

Gödel's proof employed the concept of “arithmetization of metamathematics.” He remarks:

The formulas of a formal system ... are finite sequences of primitive signs ... and it is easy to state with complete precision *which* sequences of primitive signs are meaningful formulas and which are not. Similarly, proofs, from a formal point of view, are nothing but finite sequences of formulas (with certain specifiable properties). ([1931], p. 147)

A formal system contains a countable collection of primitive signs. So a countable set of natural numbers can be assigned to the primitive signs. This device is known as “Gödel numbering.” Metamathematical propositions thus become propositions about natural numbers or sequences of them. That is, via the Gödel numbering, various predicates of natural numbers can be formulated which say things about *PM*. Hence, the metamathematical propositions can (at least in part) be expressed by the symbols of *PM* itself—they can be defined in *PM*. It is possible then to construct a proposition *A* of the system *PM* such that *A* says: “I am not provable in *PM*”.

Though it resembles the “Liar paradox” of Epimenides, *A* is not paradoxical. Every true

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<sup>7</sup> Though Russell and Whitehead succeeded in deriving a vast portion of mathematics, they were forced to accept two suspect axioms. The first, reducibility, arises in a rather ad hoc fashion, while the second, the axiom of infinity, is not purely logical in nature.

sentence need not be provable, and the technical construction of  $A$  avoids the problem of self-recursion.

Next, the proof introduces the property of consistency.<sup>8</sup> A formal system is said to be consistent if no proposition of the system can be both proved and disproved. Consistent systems guarantee that contradictory propositions or theorems cannot be derived, that is, only true propositions are provable. Gödel's proof first assumes that  $PM$  is a consistent system. Now, if  $A$  is provable in  $PM$ , then  $A$  is false, for it asserts, "I am not provable in  $PM$ ". Then by the assumption that  $PM$  is consistent,  $A$  is not provable and hence true. By the same assumption, the negation of  $A$  ( $\sim A$ ) is false, and so  $\sim A$  is also not provable. Therefore,  $A$  is undecidable in  $PM$  and  $PM$  is termed incomplete. Not all propositions defined in the formal system can be decided in the formal system. This leads to Gödel's first incompleteness theorem: If the formal system  $PM$  is consistent, then it is incomplete.

Let  $\text{Con}(PM)$  denote the proposition that  $PM$  is consistent. Then  $\text{Con}(PM)$  can be expressed as a formula within  $PM$  via the Gödel numbering. If  $\text{Con}(PM)$  were provable, then  $PM$  would be consistent and thus all true propositions in  $PM$  would be provable, including the undecidable proposition  $A$ . But from the first incompleteness theorem,  $A$  is not provable if  $PM$  is consistent. Hence,  $\text{Con}(PM)$  is not provable in  $PM$ . This is Gödel's second incompleteness theorem: If the formal system  $PM$  is consistent, then  $\text{Con}(PM)$  cannot be proved from within  $PM$ .

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<sup>8</sup> Here the term "consistency" is used in place of the more technical term "simple consistency." According to Gödel's original formulation, it was required that the system satisfy the property of  $\omega$ -consistency, a property stronger than consistency. It was later shown that  $\omega$ -consistency could be relaxed so that only consistency was required to obtain the same results. This strengthening of Gödel's theorem was accomplished by J. Barkley Rosser [1936].

The system Gödel employed was not actually *PM*, but a similar system he called *P*, though it extends to a wide variety of formal systems for mathematics or parts of mathematics. The first incompleteness theorem can be stated as saying that any consistent formal system in which a certain amount of arithmetic can be developed is incomplete. Its corollary, the second incompleteness theorem, is generally formulated as saying that the consistency of a system cannot be proved from within that system.

### REALISM, FORMALISM, AND GÖDEL'S RESULTS

Having some knowledge of the incompleteness theorems themselves, attention is now directed to the relation between Gödel's work (including the theorems) and his realism.

In response to a questionnaire put to him by Burke D. Grandjean in 1975, Gödel wrote that he was a mathematical realist since 1925, corresponding to his student days and antedating his work in mathematical logic.<sup>9</sup> On a question asking about his association with the Vienna Circle, he stated:

I don't consider my work a facet of the intellectual atmosphere of the early 20th century, but rather the opposite.<sup>[10]</sup> It is true that my interest in the foundations of mathematics was aroused by the Vienna Circle, but the philosophical consequences of my results, as well as the heuristic principles leading to them, are anything but positivistic or empiristic.<sup>11</sup>

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<sup>9</sup> Unsent response to Burke D. Grandjean's letter, Dec. 2, 1975, in *CW V*, p. 447.

<sup>10</sup> Gödel's footnote in the letter reads: "This is demonstrably true at least for the heuristic principles which led to my results (which are Platonistic)... So my work points toward an entirely different world view."

<sup>11</sup> Letter to Burke D. Grandjean, Aug. 19, 1975, in *CW V*, p. 443.

In his correspondence with Hao Wang, moreover, Gödel regularly emphasized the importance of realism to his mathematical achievements.<sup>12</sup> He states that his realist convictions provided a “heuristic principle” leading to his completeness proof of first-order predicate calculus. His “objectivist conception of mathematics and metamathematics in general, and of transfinite reasoning in particular,” was fundamental to this, and indeed all, of his work in logic. In a letter to Wang, Gödel wrote:<sup>13</sup>

The completeness theorem, mathematically, is indeed an almost trivial consequence of *Skolem 1922*. However, the fact is that, at that time, nobody ... drew this conclusion. The blindness (or prejudice, or whatever you may call it) of logicians is indeed surprising. But I think the explanation is not hard to find. It lies in a widespread lack, at that time, of the required epistemological attitude toward metamathematics and toward non-finitary reasoning.

The prominent attitude at the time was largely a product of Hilbert’s formalist program, declaring that “everything that hitherto made up mathematics is to be strictly formalized” (Hilbert [1922], p. 211). Whereas previous treatments of the foundations of mathematics (Frege’s and Dedekind’s) “proved to be inadequate and uncertain,” Hilbert advocated a different approach ([1922], p. 202):

Because I take this standpoint, the objects of number theory are for me—in direct contrast to Dedekind and Frege—the signs themselves.... The solid philosophical attitude that I think is required for the grounding of pure mathematics—as well as for all scientific thought, understanding and communication—is this: *In the beginning was the sign.*

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<sup>12</sup> Wang [1974] and [1991].

<sup>13</sup> Dec. 7, 1967, in *CW V*, 397.

Consequently, mathematics as conceived by the formalist is something of a formal game, consisting in the manipulation of meaningless signs or symbols.

The goal of Hilbert's program was to formalize all mathematical systems and then prove consistency using only finitist means. To that end, Hilbert distinguished between finitely provable and ideally provable formulas. A finitist consistency proof would depend only upon finite combinations of sign configurations and contain only "real" propositions. Ideally provable mathematics would be shown to be conservative extensions of finite mathematics. Thus, the use of meaningless infinitary statements belonging to "ideal" mathematics would be justified in terms of a finite metamathematics. As is known, however, Gödel's incompleteness theorems had significant consequences for Hilbert's elaborate formalism. The theorems demonstrated that the necessary metamathematical justification could not be carried out. A concise argument of von Neumann's, Hilbert's collaborator and once himself an advocate for formalism, offered at a meeting of the Vienna Circle on January 15, 1931, makes the point well: "If there is a finitist consistency proof at all, then it can be formalized. Therefore, Gödel's proof implies the impossibility of any [such] consistency proof."<sup>14</sup> In short, the second incompleteness theorem indicates that Hilbert's goal is unattainable.

Gödel also expressed that "another reason which hampered logicians" in making the connection between Skolem's work and the completeness proof might also be traced back to a general prejudice against realism fueled by a formalist bias. This is evidenced "in the fact that,

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<sup>14</sup> The quotation is from *Protokoll des Schlick Kreises* and was brought to my attention by Sieg [1988], p. 342. The minutes for the entire meeting are found in the Carnap Archives at the University of Pittsburgh. At the same meeting, Gödel also mentioned that it is doubtful "whether all intuitionistically correct proofs can be captured in a single formal system. That is the weak spot in Neumann's argumentation." However, though Detlefsen [1986] and others deny the applicability of Gödel's second incompleteness theorem to Hilbert's program, the theorems indicate that the program, at least as defined by Hilbert and Bernays, is unfeasible. It should be noted that Feferman [1988] has pursued a relativized version of Hilbert's program within the confines of Gödel's results.

largely, metamathematics was not considered a science describing objective mathematical states of affairs, but rather as a theory of the human activity of handling symbols.”<sup>15</sup>

Gödel’s “heuristic principle” was also at work in the development of the incompleteness results. In this case, the “principle” was specifically the highly transfinite concept of “objective mathematical truth,” as opposed to the identification (and confusion) of formal provability with mathematical truth as was common before the work of Gödel and Tarski.<sup>16</sup> Using this transfinite concept in constructing undecidable number-theoretical propositions “eventually leads to the general theorems about the existence of undecidable propositions in consistent formal systems,” that is, to the incompleteness theorems and their extension by Rosser.<sup>17</sup>

In addition, the use of the Gödel numbering device in the incompleteness proof was, though not at odds, at least considerably unnatural in a formalist scheme. Gödel notes:

How indeed could one think of *expressing* metamathematics *in* the mathematical systems themselves, if the latter are considered to consist of meaningless symbols which acquire some substitute of meaning only *through* metamathematics?<sup>18</sup>

The “arithmetization of metamathematics” was essential for the proof of the incompleteness theorems. For the realist, mathematical propositions have objective truth-values independent of metamathematics or the manipulation of symbols. There is no difficulty in expressing metamathematics in the system itself. From the formalist perspective, an interpretation of a mathematical system from within the system, according to Gödel’s model, seems “preposterous, since it is an ‘interpretation’ in terms of something which itself has no meaning” (Tieszen

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<sup>15</sup> Letter to Hao Wang, Mar. 7, 1968, in *CW V*, 404.

<sup>16</sup> Tarski [1933]. In results similar to Gödel’s, Tarski demonstrated the necessity of distinguishing between truth and provability.

<sup>17</sup> Letter to Wang, Dec. 7, 1967, in *CW V*, pp. 397-398.

<sup>18</sup> Letter to Wang, Dec. 7, 1967, in *CW V*, p. 398.

[1994], p. 187). Philosophical commitments to finitary reasoning and the meaninglessness of mathematical symbols precluded others from obtaining the kinds of results at which Gödel arrived.

However, another remark of Gödel's seems to express a contrary view of his realist convictions, and thus warrants mention as well. In a lecture Gödel delivered to the Mathematical Association of America in 1933, he devotes attention to the axiomatization of set theory and then to giving a justification for the axioms. At this point, various difficulties arise: non-constructive notions of existence, the application of quantifiers to classes, and the admission of impredicative definitions.<sup>19</sup> He follows with the surprising statement:

The result of the preceding discussion is that our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent. ([1933], p. 50)

In the text of the lecture, Gödel considers impredicativity as the most serious of the problems he mentions. He notes that specifying properties of the integers impredicatively is acceptable if we assume that “the totality of all properties [of integers] exists somehow independently of our knowledge and definitions, and that our definitions merely serve to pick out certain of these previously existing properties” ([1933], p. 50). This evidently prompts the remark that acceptance of the axioms “presupposes a kind of Platonism.”

How does the statement fit with Gödel's other assertions of his commitment to Platonism? It is possible, as Feferman and Davis have suggested, that Gödel's Platonism “regarding sets may have evolved more gradually than his later statements would suggest” ([1995], p. 40). In other words, he may have been a Platonist concerning, say, the integers, but

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<sup>19</sup> See Parsons [1995], p. 49.

not concerning sets. Another possibility is that Gödel had a “temporary period of doubt about set-theoretical Platonism” ([1995], p. 40). The options, of course, are speculative. At any rate, the philosophy seems determinative of the mathematics. Impredicative definitions are not allowed, not because they conflict mathematically, but because they conflict with certain previously-committed-to philosophical dogmas.

It is clear though that Gödel did not maintain this position for any length of time. It also illustrates that Gödel did not approach mathematical problems with the intent simply of deducing realist implications. There is the hint that intra-mathematical considerations proved decisive—in either extending his supposed initially weak Platonism or jettisoning any momentary doubts he may have had. This would mark a turn toward the idea that “the source of justification for the existence of mathematical things lies in the ordinary practice of mathematics” (Maddy [1996], p. 496). This suggestion finds support in Gödel’s other writings.<sup>20</sup> For example, in discussing Russell and Whitehead’s no-class theory, he remarks:

This whole scheme of the no-class theory is of great interest as one of the few examples, carried out in detail, of the tendency to eliminate assumptions about the existence of objects outside the “data”<sup>[21]</sup> and to replace them by constructions on the basis of these data. ([1944], p. 132)

Gödel takes the failure of the no-class theory to show that “logic and mathematics (just as physics) are built up on axioms with a real content which cannot be ‘explained away’” ([1944], p. 132). One might understand Gödel as saying that the no-class theory cannot explain the “data”, just as a “phenomenalist attempt to eliminate reference to physical objects from the

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<sup>20</sup> Maddy [1996] has suggested a similar reading and her comments have been influential to the discussion here. She does not account, however, for the “heuristic” role realism plays in Gödel’s thought.

<sup>21</sup> Gödel uses “data” to mean “logic without the assumption of the existence of classes” ([1944], p. 132, n. 33).

language of physics cannot account for its ‘data’, i.e., for sensory experience” (Maddy [1996], p. 495). While this understanding may have merit, Gödel does not say that no-class substitutes cannot account for logic. Rather, he says that they “do not have all the properties required for their use in mathematics ...” because, for instance, “... the theory of real numbers in its present form cannot be obtained” ([1944], pp. 132,134). The philosophically motivated no-class theory cannot meet the basic needs of ordinary, accepted mathematics.

As regards Russell’s vicious circle principle,<sup>22</sup> Gödel notes that “it is demonstrable that the formalism of classical mathematics does not satisfy the vicious circle principle in the first form, since the axioms imply the existence of real numbers definable in this formalism only by reference to all real numbers.” Gödel’s conclusion is clear: “I would consider this rather as a proof that the vicious circle principle is false than that classical mathematics is false ...” ([1944], p. 127). It is because the vicious circle principle (VCP) prohibits the derivation of classical mathematics that it is false.

What stands out here ... is that the argument does *not* run: the VCP is an anti-realist claim; realism is correct for reasons  $x, y, z$ ; therefore, the VCP is false.

Rather, the argument goes straight from mathematical actualities to the falsity of the VCP, without a detour through any extra-mathematical theorizing about the nature of mathematical things. That theorizing only begins *after* the above conclusion has been drawn. (Maddy [1996], p. 496)

After stating his conclusion, Gödel goes on to say that the falsity of the vicious circle principle is “indeed plausible also on its own account” ([1944], p. 127). The principle, he notes, only applies if one takes a “constructivistic (or nominalistic) standpoint toward the objects of logic and

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<sup>22</sup> The vicious circle principle is a claim about the properties of collections that requires eschewing impredicative specifications.

mathematics” ([1944], p. 128). It is at this point that Gödel contends for realism over against “constructivistic” or “nominalistic” attitudes.

This trend in Gödel’s thought continues in his article on Cantor’s continuum problem [1964]. The concern is whether or not the question of the continuum hypothesis would continue to have meaning should it come out to be independent of ZFC (Zermelo-Fraenkel set theory plus the axiom of choice):<sup>23</sup>

... a proof of the undecidability of Cantor’s conjecture from the accepted axioms of set theory ... would by no means solve the problem. For if the meanings of the primitive terms of set theory ... are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor’s conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality. Such a belief is by no means chimerical, since it is possible to point out ways in which the decision of a question, which is undecidable from the usual axioms, might nevertheless be obtained. ([1964], p. 260)

Both intra-mathematical and extra-mathematical considerations are expressed. For the realist, there is a real world of sets in which Cantor’s conjecture is either true or false. Thus, it remains a meaningful question. This is the philosophical argument for the meaningfulness of the continuum hypothesis. That such a position is “by no means chimerical” comes from practical mathematical concerns and indeed Gödel goes on to explore how the axioms of set theory can be naturally extended and how somewhat unnatural axioms can be accepted on the basis of their

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<sup>23</sup> Gödel had previously shown that the continuum hypothesis was consistent with ZFC. See Gödel [1940]. Paul Cohen [1963] finally proved that the problem was independent of ZFC.

mathematical consequences. Again, the commitment to realism provides one line of reasoning. The possibilities for solving the continuum hypothesis on the basis of new axioms, a concern coming from within mathematics, represent another line of reasoning. This second line upholds the first. Mathematical concerns provide the foundational argument for realism.<sup>24</sup> In other words, the mathematics supports the philosophy (Maddy [1996], p. 497).

## INTUITIONISM

In discussing the relation between Gödel's mathematical work and his realism, attention has already been called to the school of formalism. Now, focus is directed toward the other two major schools of thought—intuitionism and logicism—beginning with the former.

Research in intuitionist and constructivist mathematics spans a broad spectrum. Building on constructivist themes quite explicitly Kantian,<sup>25</sup> the tenets of intuitionism gained newfound clarity in the early-twentieth century. As it matured, the field included the intuitionist “mediating” postures taken by Poincaré (towards classical mathematics)<sup>26</sup> and the later Hermann Weyl (towards formalism).<sup>27</sup> The range continued, from Borel and the French school,<sup>28</sup> over to Heyting,<sup>29</sup> and finally to its chief proponent, namely, Brouwer and his intuitionist program.<sup>30</sup>

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<sup>24</sup> Parsons ([1990], p. 107) points out that “Gödel saw his realism in the context of concrete problems and as motivating mathematical research programs.”

<sup>25</sup> Kant's discussion of space and time is particularly germane. See Kant [1933], A22\B37-A42\B59, p. 67-82; Posy [1984]; and Brouwer [1912].

<sup>26</sup> See Folina [1992], pp. 172-192.

<sup>27</sup> See Mancosu [1998].

<sup>28</sup> Borel [1967]. On the French Intuitionists, see Largeault [1993] and [1993a].

<sup>29</sup> Heyting [1930], [1931], and [1966].

Newer approaches, while branching off in significant places, all trace back in some way to Brouwer.

In the 1912 inaugural address at the University of Amsterdam, Brouwer provided a basic yet insightful characterization of his ideas, making an explicit distinction between his position and formalism as he commented on mathematics as *the* “exact” science: “The question where mathematical exactness does exist is answered differently by the two sides; the intuitionist says: in the human intellect; the formalist says: on paper” ([1912], pp. 81-82). After marking out some of intuitionism’s historical roots, he states:

From the present point of view of intuitionism therefore all mathematical sets ... can be developed out of the basis of the basal intuition.... And in the construction of these sets neither the ordinary language nor any symbolic language can have any other role than that of serving as a non-mathematical auxiliary, to assist the mathematical memory or to enable different individuals to build up the same set. ([1912], p. 85)

Clearly, the formalization sought by Hilbert is decidedly not a task that Brouwer undertakes.

Intuitionism differs at other points as well. Heyting relates that “intuitionistic mathematics is a mental activity” ([1930], p. 311). Again following Brouwer, traditional intuitionism conceives of mathematics as pure, individual thought-construction: mathematics is a free creation, “... created by a free action” ([1907], p. 179). Mathematical existence, strictly considered, is “having been constructed.” A mathematical statement is true only if one is in possession of a proof of it, and false only if one is in possession of a refutation. Concerning the future, the adopted attitude is neutralism. In fact, “Rejection of the principle of bivalence for statements of some given class always involves a repudiation of a realistic interpretation of them;

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<sup>30</sup> Brouwer [1928], [1929], and [1930]. Also see van Stigt [1990].

and adoption of an anti-realistic view often turns critically upon such a rejection of bivalence” (Dummett [1982], p. 231).

Intuitionism answers ontological questions about mathematical objects differently than formalism, which views the objects as meaningless symbols. Rather, intuitionism says that the objects do exist, but only insofar as they are conceived as “being essentially the result of a mathematical activity that produces them” (Bouveresse [2005] p. 60). The properties of mathematics are “determined by the time-bound and individual nature of mind as the sole creator and seat of mathematical thought” (van Stigt [1998] p. 7), and not by the nature of the mathematical objects themselves. This is also clearly at odds with realism over the mind-independence of mathematical objects.

Furthermore, characteristic of the position is the intuitionist campaign to reform classical mathematics—the reductive thesis. Perhaps most notable is the rejection of the logical law of excluded middle, or *tertium non datur* (i.e., a third is not given), which states that the disjunction of a statement with its negation is always true.<sup>31</sup> Brouwer pronounced *tertium non datur* *not permissible* as part of mathematical proof in order to claim support for his intuitionist set theory (Brouwer [1921], p. 198). His set-theoretic proposal would have severe consequences not only for classical mathematics but also for Hilbert’s own formalism. For these reasons, Hilbert offered the rejoinder:

What Weyl and Brouwer do amounts in essence to following the erstwhile path of Kronecker. They seek to ground mathematics by throwing overboard all phenomena that make them uneasy.... This means, however, to ... mangle our

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<sup>31</sup> Bivalence and the law of excluded middle are similar, but not the same. The principle of bivalence states that a proposition  $P$  must be either true or false, e.g., the sentence, “The girl is married” must be either true or false. The law of excluded middle states that  $(P \text{ or } \sim P)$  is true, e.g., the sentence, “Either there is a tree that is over 400 feet tall or it is not the case that there is such a tree” is true. While Dummett refers to intuitionism’s rejection of bivalence ([1982], pp. 310-312), Brouwer initially campaigned against the law of excluded middle.

science, and if we follow such reformers, we run the danger of losing a large number of our most valuable treasures. ([1922], p. 23)

Lest Hilbert's statement be construed as the overreaction of one embroiled in the formalist-intuitionist controversy of the day,<sup>32</sup> Weyl is even more forthright in his estimation of the difficulty of the Brouwer program and its inevitable conclusion:

With Brouwer, mathematics gains its highest intuitive clarity. He succeeds in developing the beginnings of analysis in a natural manner.... It cannot be denied, however, that in advancing to higher and more general theories the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the larger part of his towering edifice, which he believed to be built of concrete blocks, dissolve into mist before his eyes. ([1949], p. 54)

The consequences of accepting the intuitionist reform are really more akin to revolution, since the pieces of classical mathematics not rejected must be profoundly revised. More contemporary philosophers of intuitionism acknowledge the drive for reconstruction and revision as well. Dummett, whose recent work arguably presents the most formidable case for intuitionism as a theory of mathematics, deems philosophy of language to take such precedence over classical mathematics and logic that, to the point where, should differences arise, the mathematician ought to change his ways ([1977], p. 377).

In echoing an important objection to intuitionism, disallowing portions of mathematics because they fail to conform to a particular philosophical position raises serious concerns. How much of a critiquing-function is one to grant the philosophy of mathematics? In the philosophy

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<sup>32</sup> For a synopsis of the controversy between Hilbert and Brouwer, see Reid [1970], pp. 184-188 and van Stigt [1998], pp. 1-3.

of science, for example, the task is, generally, to increase understanding of science as it is practiced, as opposed to producing a new kind of science. It is doubtful that intuitionism can boast of such strong support that it would be most reasonable to alter logic and mathematics in so radical a fashion. It would be more sensible to revisit the philosophical theory rather than cast off extensive portions of classical mathematics. This objection, of course, is in keeping with Gödel's own argumentation. Mathematics should not be unnecessarily sacrificed to accommodate philosophical speculations. For instance, while a Platonist is free to admit non-classical logical operations, the reductive thesis prevents the intuitionist from admitting classical ones. Gödel's realism was supported by his work in mathematics. Intuitionism, however, is at variance with standard mathematical methodology from the beginning.

Gödel's first incompleteness theorem may suggest other difficulties for intuitionism. To develop this, it is first helpful to recall what distinguishes intuitionism from other perspectives: rejection of bivalence (and the law of excluded middle), both impredicative and other non-constructive proofs and definitions, and the depiction of mathematics as pure thought-construction. With this as background, a version of Gödel's first theorem may be read in the following way:

The first incompleteness theorem suggests that the abstract concept of objective arithmetic truth transcends our intuition (or constructive abilities) at any given stage, in the sense that we know we can always construct additional instances of this concept at various times that we have not yet intuited or constructed, in the form of the specific Gödel sentences  $[A', A'', A''', \dots]$ . The concept of arithmetic truth then appears to be known as an identity (or "universal") through these differences which "transcends" the construction of the specific instances at

any given stage in time. This identity (or “universal”) is “outside of” or “independent of” each particular intuition (construction) ... [and] this is just how we characteriz[e] realism ...<sup>33</sup>

As traditional intuitionism does not allow for mathematics or its characteristics to lie outside of one’s constructions, it is unclear how this concept of arithmetic truth fits into the intuitionist scheme. Due to the incompleteness of a consistent formal system (Gödel’s first theorem), the concept of objective arithmetic truth that “transcends” our constructions as outlined certainly seems agreeable—a conclusion it appears that intuitionism does not arrive at.

More sophisticated forms of constructivism may or may not be able to accommodate this concept. If they did prove compatible, they would at least be moving beyond traditional intuitionism. Perhaps at this point it should be remarked that just as the term “Platonism” as applied to mathematics has taken on (and discarded) different connotations, at least since its use by Bernays,<sup>34</sup> not all positions going under the banner of intuitionism or constructivism are equal. It seems, though, that should new forms of constructivism prove compatible, the required move beyond Brouwer would also be a move in the direction of the realist. But even now, upon this reading, the first incompleteness theorem seems to favor mathematical realism.

## LOGICISM AND CONVENTIONALISM

The position of Logicism in the philosophy of mathematics is now considered. Frege was the first to substantially develop the principal thesis of the logicist school, that analysis and

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<sup>33</sup> From Tieszen [1994], p. 188. Tieszen is considering the first incompleteness theorem in the context Dedekind-Peano Arithmetic.

<sup>34</sup> See Bouveresse [2005] and Bernays [1935].

arithmetic can be reduced to logic.<sup>35</sup> Russell and Whitehead carried on the proposal in *Principia*, though as has been seen, the incompleteness theorems disrupted their efforts. Another position known as conventionalism, largely a revived and improved form of logicism though not identical to Frege's or Russell's, emerged from Wittgenstein,<sup>36</sup> the Vienna Circle, and with its greatest sophistication, Rudolf Carnap's *Logical Syntax of Language* [1937].<sup>37</sup>

The Positivists of the Vienna Circle took care in distinguishing between synthetic, factual, empirical truths corresponding to reality and analytic truths, holding regardless of how the facts lie. Thus, one finds statements of the following in *Logical Syntax*:

The investigation will not be limited to the mathematico-logical part of the language ... but will be essentially concerned also with synthetic, empirical sentences. The latter, the so-called "real" sentences, constitute the core of science; the mathematico-logical sentences are analytic, with no real content, and are merely formal auxiliaries ... ([1937], p. xiv)

and,

... an analytic sentence is absolutely true whatever the empirical facts may be.

Hence, it does not state anything about facts.... A synthetic sentence is sometimes true—namely, when certain facts exist—and sometimes false; hence it says something as to what facts exist. *Synthetic sentences* are the *genuine statements about reality*. ([1937], p. 41)

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<sup>35</sup> Frege [1884].

<sup>36</sup> Wittgenstein [1921]

<sup>37</sup> Carnap states, "Wittgenstein's view is represented, and has been further developed, by the Vienna Circle, and in this part of the book [*Logical Syntax*] I owe a great deal to his ideas. If I am right, the position here maintained is in general agreement with his, but goes beyond it in certain important respects" ([1937], p. 282).

Analytic sentences are true by virtue of meaning. Synthetic sentences are true by virtue of the way the world is.

In Carnap's terminology, a language or a linguistic framework provides the context for all rational inquiry and discourse. The linguistic framework specifies the logical relations of contradiction and consequence. There are alternative frameworks, alternative languages, and alternative logics. This culminates in Carnap's central doctrine:

*In logic, there are no morals.* Everyone is at liberty to build up his own logic, i.e., his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments. ([1937], p. 52)

And thus the pluralism of the "*Principal of Tolerance*: It is not our business to set up prohibitions, but to arrive at conventions" ([1937], p. 51). Mathematical truths then are the consequences of adopting a particular linguistic framework. They are analytic, true by virtue of the conventions of the chosen language. The goal of this "syntactical viewpoint" is to completely reduce mathematics to syntax of language, so that the "validity of mathematical theorems consists solely in their being consequences of certain syntactical conventions about the use of symbols, not in their describing states of affairs in some realm of things" (Gödel [1953/9], version III, p. 335).

The analytic/synthetic distinction suffered, of course, from Quine's analysis in his renowned "Two Dogmas of Empiricism." He concludes:

It is obvious that truth in general depends on both language and extralinguistic fact. The statement "Brutus killed Caesar" would be false if the world had been different in certain ways, but it would also be false if the word "killed" happened

rather to have the sense “begat.” Hence the temptation to suppose in general that the truth of a statement is somehow analyzable into a linguistic component and a factual component. Given this supposition, it next seems reasonable that in some statements the factual component should be null; and these are the analytic statements. But, for all its a priori reasonableness, a boundary between analytic and synthetic statements simply has not been drawn. That there is such a distinction to be drawn at all is an unempirical dogma of empiricists, a metaphysical article of faith. ([1951], pp. 36-37)

The collapse of the dogma of some breach between analytic and synthetic statements is notable. Yet, as Quine’s paper has garnered the greater part of commentators’ attention, another important criticism of the Vienna Circle and its linguistic or syntactical account of mathematics has gone largely overlooked.

In an unpublished paper entitled “Is Mathematics Syntax of Language?” [1953/9], Gödel develops an argument based on his second incompleteness theorem. Though principally directed toward Carnap, Gödel makes his intentions plain from the beginning: “I am not concerned in this paper with a detailed evaluation of what Carnap has said about the subject, but rather my purpose is to discuss the relationship between syntax and mathematics from an angle which, I believe, has been neglected” ([1953/9], version III, pp. 335-336, n. 9).

The picture Gödel suggests is this. In order to be justified in using a particular language when reasoning about empirical concerns, there must be some reason to believe that the syntactical rules specifying the consequence relation “do not imply the truth or falsehood of any

proposition expressing an empirical fact” (Gödel [1953/9], version V, p. 357).<sup>38</sup> For this condition to be met, the rules of syntax must be consistent. Without consistency, the rules imply all propositions, factual ones included. On the basis of the second incompleteness theorem, in order to prove consistency—to legitimize the rules of syntax—mathematics outside of the rules of syntax must be used. Therefore, conventionalism’s claim that all of mathematics is a result of certain syntactical stipulations is contradicted.

Ricketts, among others, has argued that Gödel employs a “language-transcendent notion of empirical fact or empirical truth” ([1994], p. 180). First there are empirical/factual sentences, true or false by virtue of the way the world is. Conventionally stipulated analytic sentences (i.e., mathematics) are then added. The addition must be known not to affect the given empirical sentences. To know this, however, requires additional mathematics, invalidating the conventionalist scheme. Yet, Ricketts contends, “This notion of empirical fact imposes morals in logic on the conventionalist. Carnap, in adopting the principle of tolerance, rejects any such language-transcendent notions” ([1994], p. 180). For Carnap, in order to make sense of such notions as the “factual” or “empirical world,” one must already have in place a linguistic framework with its rules of language and mathematics.

Throughout *Logical Syntax*, Carnap does not worry over consistency proofs.<sup>39</sup> The adoption of an inconsistent framework is in keeping with the Principle of Tolerance. As he notes in the foreword to *Logical Syntax*:

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<sup>38</sup> A similar statement in version III reads: “A rule about the truth of sentences can be called *syntactical* only if it is clear from its formulation, or if it somehow can be known beforehand, that it does not imply the truth or falsehood of any ‘factual’ sentence” ([1953/9], p. 339).

<sup>39</sup> To be sure, Carnap was well aware of the consequences of the second incompleteness theorem for consistency proofs. He does provide a consistency proof for his Language II, but notes that “since the proof is carried out in a syntax-language which has richer resources than Language II, we are in no wise guaranteed against the appearance of contradictions in this syntax-language, and thus in our proof” ([1937], p. 129). The consistency of Language II

The first attempts to cast the ship of logic off from the *terra firma* of the classical forms were certainly bold ones, considered from the historical point of view. But they were hampered by the striving after “correctness”. Now, however, that impediment has been overcome, and before us lies the boundless ocean of unlimited possibilities. ([1937], p. xv)

Carnap’s characterization of “liberty” entails “freedom” from the insistence that “deviations must be justified—that is, that the new language-form must be proved ‘correct’” ([1937], p. xiv). Of course, an inconsistent framework would be profoundly inadequate as a foundation for mathematics. From Carnap’s position, the issue is one of pragmatics.

On first glance it may seem that Gödel has indeed missed the liberality of the Principle of Tolerance. Carnap does not demand consistency of his framework and so cannot be faulted for his non-foundationalist approach to mathematics. Gödel’s criticism, however, is not simply about providing a foundation. Certainly Carnap succeeds in giving an explication of mathematics as syntax of language, but, Gödel notes, under such an explication “mathematics remains a mystery” (Crocco [2003], p. 22). Gödel recognizes the issue of pragmatics and takes it up as well.

Suppose, Gödel says, we adopt certain rules of syntax to replace our notions of truth and consequence in mathematics, granting also that the rules need not be demonstrably consistent. Can we then affirm that we have given an entirely linguistic explanation of mathematics, as conventionalism purports to do? Even if our rules are inconsistent and cannot provide a foundation for mathematics, they do permit us to clearly describe the linguistic nature of mathematics—we have an appropriately clear language or linguistic framework. Our

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can only be proven in a richer metalanguage, and the consistency proof for this metalanguage requires in turn a richer language, and so the hierarchy of languages becomes stronger and stronger.

mathematical activities now are to be explained through notions of convention. This should lead to the same sorts of conclusions which can be deduced using mathematics. In other words, the linguistic explication of mathematics “should not destroy our trust in the predictive power of mathematics, predictive power both for pure mathematics and for applied mathematics” (Crocco [2003], p. 35). For instance, we can predict certain properties about the integers using mathematical theorems. Also, using the “physical theory of elastic body, which can be formulated only using a certain portion of mathematics, we can predict whether a certain bridge, constructed according to these laws, falls down or not.” Whatever trust we might have in these predictions would be misplaced if the rules which allow us to formulate these predictions were “simple conventions without content” (Crocco [2003], p. 35). Gödel concludes that “the scheme of the syntactical program to replace [mathematical activities] by rules for the use of symbols fails because this replacing destroys any reason for expecting consistency, which is vital both for pure and applied mathematics” ([1953/9], version III, p. 346). The concern is a practical one.

Providing a linguistic explanation of mathematics (without explaining mathematical applicability) is at least “as odd as an explanation or a description of soccer rules which ... omit the fact that the players play to win and to put the ball inside the net” (Crocco [2003], p. 36). It is odd to say that when explaining the nature of mathematics it is unnecessary to also explain its applicability. Such an approach is, nevertheless, coherent. But from Gödel’s perspective, the liberality of the conventionalist program renders it useless. He observes:

In general, if by using concepts and their relations among each other and with the sensations, one arrives at verifiable consequences, it is exactly from the existence of the objects having these relations that the verifiable consequences follow....

That the existential assertions, also in mathematics, are not mere “façon de parler”

follows from the fact that they can be disproved (by inconsistencies derived from them) and that they have consequences as to ascertainable facts.... There is no reason to answer the question of the objective existence of mathematical and space-time objects differently. ([1953/9], version III, pp. 355-356)

As for pragmatics:

If an inconsistency is not recognized as a disproof, but only as a proof for the “inexpediency” of the “convention” in question, the same can be done for laws of nature, which also can be interpreted to be conventions which become “inexpedient” in case a counterexample is met with. ([1953/9], version IV, p. 361)

Conventionalism thus has consequences affecting more than just mathematics. An anti-realist position on mathematics may in fact lead to a broader anti-realism about chemistry, physics, and other sciences.

It is not claimed that this reading of Gödel’s position on Carnap’s conventionalism has no difficulties. Gödel’s lack of concern for the particular details of Carnap’s proposal in *Logical Syntax* may arguably affect his assessment of it, but this picture does provide a useful perspective and highlights Gödel’s account of the meaningfulness of mathematics. In the least, Gödel’s criticisms based on the incompleteness results and the desire to explain the usefulness of mathematics cannot be as easily dismissed as Ricketts seems to do.

On all these points concerning formalism, intuitionism, and logicism, it seems that dividends result from being cautious about imposing foreign philosophical restrictions on mathematics.

It makes sense, for instance, to hold that “mathematical truth” may not be reducible to “formal provability” or even to “intuitionistic provability”. That is, it should not be assumed from the outset that these expressions have the same meaning, as if they are related analytically, but we should instead see what happens as they are further clarified. Thus, there are some grounds for a reverse skepticism about philosophical views that tend to be eliminativist about basic notions of logic and mathematics.... They might actually hinder *mathematical* progress at the expense of some philosophical theory. (Tieszen [1994], p. 189)

Having traversed the major schools of thought in the philosophy of mathematics and noting some of the difficulties Gödel’s work presents for them, the relation between Gödel’s intra-mathematical and extra-mathematical concerns for realism is examined now in connection with Gödel’s argument against the view that mathematics is, in some sense, our own creation.

#### THE GIBBS LECTURE (GÖDEL’S DISJUNCTIVE CONCLUSION)

In December of 1951 Gödel delivered the twenty-fifth Josiah Willard Gibbs lecture, “Some Basic Theorems on the Foundations of Mathematics and their Implications” [1951] before a gathering of the American Mathematical Society, an address presenting what may be Gödel’s most detailed argument for realism. The main concern of the Gibbs lecture is to explore certain logical results and what Gödel considers to be their philosophical consequences—thus far, the familiar pattern from mathematics to philosophy.

The starting place is what he describes as the phenomenon of the inexhaustibility or incompleteness of mathematics. This phenomenon is explained from two different angles. First

is an attempted axiomatization of set theory: begin with the integers, iterate the power-set operation through finite ordinals, and formulate axioms to describe the sets formed in the process. The sequence of operations to which this iterative procedure is applied has no end, so there is also no end to the formation of axioms; "... nor can there ever be an end to *this* procedure of forming the axioms, because the very formulation of the axioms up to a certain stage gives rise to the next axiom" ([1951], p. 307). Other comments by Gödel on the iterative concept of set, as recorded by Wang,<sup>40</sup> indicate that the axioms formulated imply all the axioms of ZF plus the axioms of replacement. This axiomatic approach to set theory has no end and thus illustrates the inexhaustibility of mathematics.

Yet regardless of one's approach to the foundations of mathematics (if, for instance, it is not the axiomatization of set theory), Gödel claims that the conclusion that mathematics is inexhaustible still follows. The second explanation he gives of the phenomenon is "entirely independent of the special standpoint taken toward the foundations of mathematics" ([1951], p. 310), and so arrives at the same end.

This second and more significant path to incompleteness is through the incompleteness theorems. After remarking that the first incompleteness theorem is "equivalent to the fact that there exists no finite procedure for the systematic decision of all Diophantine problems of the type specified" ([1951], p. 308), little further reference to the first theorem occurs.<sup>41</sup> It is the second theorem which attracts most of Gödel's attention in his discussion of the incompleteness of mathematics.

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<sup>40</sup> See Wang [1974]

<sup>41</sup> The Diophantine problems in view are to determine the truth or falsity of sentences of the form, "For every  $x$ , there exists  $y$  such that  $P(x, y) = 0$ , where  $x$  and  $y$  are sequences of integer variables and  $P(x, y)$  is a polynomial with integral coefficients." See *CW III*, pp. 307-309 and Boolos [1995], p. 292.

Now the theorem says that for any well-defined system of axioms and rules, in particular, the proposition stating their consistency (or rather the equivalent number-theoretical proposition) is undemonstrable from these axioms and rules, provided these axioms and rules are consistent and suffice to derive a certain portion of the finitistic arithmetic of integers. ([1951], pp. 308-309)

Accordingly, as Gödel sees it, it is this theorem which makes the incompleteness of mathematics particularly evident.

For, it makes it impossible that someone should set up a certain well-defined system of axioms and rules and consistently make the following assertion about it: All of these axioms and rules I perceive (with mathematical certitude) to be correct, and moreover ... they contain all of mathematics. ([1951], p. 309)

The impossibility of such an assertion is a direct result of the second incompleteness theorem.

“If someone makes such a statement he contradicts himself. For if he perceives the axioms under consideration to be correct, he also perceives (with the same certainty) that they are consistent. Hence he has a mathematical insight not derivable from his axioms” ([1951], p. 309).

On account of the second theorem, the consistency of the axioms of some formal system cannot be proved from within that system. Therefore, it cannot be claimed that the system contains all of mathematics.

In order to prevent possible misinterpretations, Gödel clarifies that no well-defined system of correct axioms can contain all of what he terms objective mathematics, the system of all true mathematical propositions. Mathematics in the subjective sense, the system of all demonstrable mathematical propositions, is not affected. But if a finite rule producing all of subjective mathematics did exist, we would never know if it was correct. Only one proposition

after another, for any finite number of them, could be perceived to be true. The proposition that they were all true could be asserted only with appeal to some inductive inference, without of course, mathematical certainty.

If such a rule did exist, it would mean that the human mind is “equivalent to a finite machine that, however, is unable to understand completely its own functioning” ([1951], p. 310), for the insight that the mind produces only “correct (or only consistent) results would surpass the powers of human reason” ([1951], p. 310, n. 14). Purportedly a (consistent) machine that “completely understands” its own functioning would thus comprehend its consistency. But the assertion of consistency can be expressed, via Gödel numbering, as a sentence in the class of sentences made up of the above Diophantine problems. Therefore, Gödel holds, the following disjunctive conclusion is inevitable:

Either mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable Diophantine problems of the type specified ... ([1951], p. 310)

The first incompleteness theorem is “equivalent to the fact that there exists no finite procedure for the systematic decision of all” simple problems about the natural numbers. If the human mind does not surpass the powers of a finite machine, a Turing machine for instance, then by the first theorem, there exist such problems which are undecidable by the human mind.<sup>42</sup> Notably,

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<sup>42</sup> In one of the few works that take notice of Gödel’s disjunctive conclusion, Salmon remarks that “Gödel’s principal argument does not make any essential detour through Turing machines, or machines of any sort. One can dispense with machines altogether and make an end run for a disjunctive conclusion of just the sort from which Gödel draws philosophical conclusions about the mind and the objectivity of mathematics” ([2001], p. 98). Salmon’s reconstruction, however, seems to introduce unnecessary complications.

Gödel refers to the conclusion as a mathematically established fact and of great philosophical interest ([1951], p. 310).<sup>43</sup>

It is not exactly clear what Gödel means as he suggests the possibility that the human mind is “equivalent to a finite machine” ([1951], p. 310). Salmon notes, “The difficulty of likening the theorem-proving capacity of the human brain to a computer is not so much that the brain’s cognitive processes are mechanistic. Nor is it that a machine cannot know the fundamental axioms of human [i.e., subjective] mathematics.” Instead, the difficulty comes “in the very design (let alone the construction) of a theorem-proving machine when there is no effective procedure for delimiting its proofs’ admissible starting points.” This means that either there is no such procedure with “regard to the human mind’s capacity for attaining knowledge with mathematical certainty in pure mathematics, or else there are purely mathematical problems of a certain sort that are in principle unsolvable by the human intellect. This is Gödel’s disjunction” (Salmon [2001], p. 104). In any case, though the following statement is vague, it does appear to be a consequence of the second incompleteness theorem: No theorem-enumerating Turing machine can print an output synonymous to an assertion of its own consistency.<sup>44</sup> That is to say, any mind whose theorem-proving capacity can be represented by some Turing machine is in principle incapable of solving certain mathematical problems.

According to Wang ([1974], pp. 324-326), Gödel was in agreement with Hilbert’s rejection of the existence of absolutely undecidable problems, and so preferred the first

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<sup>43</sup> Gödel’s disjunctive conclusion is not the same as the conclusion drawn by Nagel and Newman [1959], J.R. Lucas [1961] and [1996], or Roger Penrose [1989] and [1994], who contend that the theorems directly imply that a Turing machine cannot serve as a model for the human mind. In effect, they appear to argue thus: Let  $T$  be a Turing machine that “represents” my own mathematical abilities. Then applying Gödel’s technique, it is possible for me to find a proposition that I can prove, but  $T$  cannot prove. This contradicts the statement that  $T$  “represents” me. Therefore, the mind is not equivalent to a Turing machine. Putnam [1960] has given a straightforward response to this argument: To find some proposition  $U$  from an arbitrary machine  $T$  such that, if  $T$  is consistent,  $T$  cannot prove  $U$  is not in fact to prove  $U$ . Gödel’s more modest conclusion does not suffer from this flaw.

<sup>44</sup> See Salmon [2001], pp. 98, 113, n. 10 and Boolos [1995], p. 293.

alternative. Perhaps following Kant,<sup>45</sup> both maintained that human reason would be irrational if it would ask questions it could not answer while asserting that only reason could answer them. Evidently, Gödel was inclined to accept the first alternative for additional reasons independent of his rejection of absolutely undecidable problems. His later writings, in particular remark 3 of [1972], give arguments against Turing's view that "mental procedures cannot go beyond mechanical procedures."

Gödel observes that the consequences to be drawn from either alternative of the disjunctive conclusion are "very decidedly opposed to materialistic philosophy" ([1951], p. 311). If the first alternative holds, so that the operations of the human mind cannot be reduced to those of the brain (a finite machine with a finite number of parts—the neurons and their connections), then, he argues, some form of vitalism is unavoidable. Of course, Gödel recognizes, "It is not known whether the first alternative holds, but at any rate it is in good agreement with the opinions of some of the leading men in brain and nerve physiology," who deny the possibility of a purely mechanistic explanation of mental processes ([1951], pp. 311-312).

Some, however, perhaps influenced by the idea of the mind as a Turing machine, might assume a more modest picture and find little troublesome with the proposition that there are mathematical truths whose proofs transcend the comprehending capabilities of the human mind. It does not seem too outlandish to suggest the possibility that the second alternative holds. Not finding Gödel's reasons for rejecting the idea compelling, one might think, why should there not be absolutely undecidable mathematical problems? Accordingly, one adopts the second alternative. But this too has important consequences.

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<sup>45</sup> As Kant states, "There are sciences the very nature of which requires that every question arising within their domain should be completely answerable in terms of what is known, inasmuch as the answer must issue from the same sources from which the question proceeds" ([1933], A476/B504, p. 430). He goes on to include pure mathematics under this heading ([1933], A480/B508, p. 433).

The significance of the second alternative of the disjunctive conclusion, that “there exist absolutely undecidable mathematical propositions,” Gödel contends, is that it seems to disprove the view that mathematics is only our own creation. As a result, this alternative entails that “mathematical objects and facts (or at least *something* in them) exist objectively and independently of our mental acts and decisions, that is to say, it seems to imply some form or other of Platonism or ‘realism’ as to the mathematical objects” ([1951], pp. 311-312). Gödel’s reason for arriving at the conclusion that mathematics is not our own creation from the second alternative is straightforward: “For the creator necessarily knows all the properties of his creatures, because they can’t have any others except those he has given to them” ([1951], p. 311). Of course, the proposal that a creator necessarily knows all the properties of his creatures is not uncontroversial.

In the first place, one might very well object that a creator need not necessarily know each and every property of what he creates. On this point Kreisel comments:

I do not make the assumption that, if mathematical objects are our own constructions, we must be expected to be able to decide all their properties; for, except under some extravagant restrictions on what one admits as the self I do not see why one should expect so much more control over one’s mental products than over one’s bodily products—which are sometimes quite surprising. ([1967])

For instance, we build machines and still cannot entirely predict their behavior. But, from Gödel’s perspective, “This objection is very poor, for we don’t create the machines out of nothing, but build them out of some given material. If the situation were similar in mathematics, then this material of basis for our constructions would be something objective,” so that, even “if

certain other ingredients of mathematics were our own creation,” all of mathematics would not be. At least some kind of mathematical reality cannot be explained away.

Other possible points of contention remain. Consider a proposition about all integers. Since it is impossible to verify the proposition for all integers one by one, it may be contended that the proposition’s meaning consists only in the display of a general proof. Suppose that is an undecidable proposition about all integers under consideration. Then both the proposition and its negation are not true. Hence, the proposition and its negation do not convey some objectively existing but unknown property of the integers. Gödel gives a twofold response to this objection.

With regard to epistemology, it “certainly looks as if one must *first* understand the meaning of a proposition *before* he can understand a proof of it, so that the meaning of ‘all’ could not be defined in terms of the meaning of ‘proof’” ([1951], p. 313). It is unclear whether or not this reply can be consistently maintained. Even so, one may surmise the truth of a universal proposition (for example, that one may be able to verify a certain property for any given integer) and at the same time conjecture that no general proof for this fact exists ([1951], p. 313, n. 19):

For the first half of it, this would ... be the case if the proposition in question were some equation  $F(n) = G(n)$  of two number-theoretical functions which could be verified up to *very* great numbers  $n \dots$  [For example] the probability of the proposition which states that for each  $n$  there is at least one digit  $\neq 0$  between the  $n$ -th and  $n^2$ -th digits of the decimal expansion of  $\pi$  converges toward 1 as one goes on verifying it for greater and greater  $n$ .

The abhorrence of mathematicians to such inductive arguments in mathematics may be “due to the very prejudice that mathematical objects somehow have no real existence. If mathematics

describes an objective world ... there is no reason why inductive methods should not be applied in mathematics” ([1951], p. 313). Incidentally, this interesting reply of Gödel’s bears similarities to empiricism. This does not mean that Gödel sees no need for a proof of, say, Goldbach’s conjecture. Instead, what he suggests is that positing an “objective reality” of mathematical entities (and consequently giving inductive arguments) is not so different from making similar claims concerning the objective physical world ([1951], p. 313, n. 20).

The consequences drawn under the second alternative, favoring realism, are supported by developments in the foundations of mathematics—developments independent of both alternatives of Gödel’s disjunctive conclusion. Gödel makes three particular remarks. First of all, he argues, a high degree of clarity has been attained in the foundations of mathematics. But this attainment of clarity has not helped much in deciding mathematical problems. This circumstance would be highly unlikely if mathematics were in any sense our “free creation.” For, “If mathematics were our free creation, ignorance as to the objects we created, it is true, might still occur, but only through lack of a clear realization as to what we really have created,” or to computational complexity ([1951], p. 314). At least in principle, our ignorance would have to disappear once we attained perfect clarity.

At this point it may not be plain that perfect clarity about our creation yields perfect knowledge of it. “Mathematics might be our own creation and we might have attained perfect clarity with respect to all the fundamental properties of what we have created, but we might nevertheless be rather ignorant about non-fundamental properties” (Boolos [1995], p. 297). Yet if we understand the foundations of mathematics with *perfect* clarity, it seems entailed that we would further understand the properties, both fundamental and non-fundamental, of our mathematical creations.

In fact, Gödel is simply referring to the reality that work on the foundations of mathematics has not done much to help mathematicians prove theorems. Suppose a group of number theorists and experts in partial differential equations (PDEs) are gathered together and told all about the axioms of ZFC, the subtleties of predicate calculus, and, say, Gentzen's work on Peano Arithmetic.<sup>46</sup> Most likely they will not prove more number theory and PDE theorems as a result. If mathematics were merely a game we created like chess, then being fully knowledgeable about its foundations would make one a better mathematician than otherwise, just as being fully knowledgeable about the rules of chess makes one a better chess player than otherwise.

Gödel next calls attention to the fact that in order to demonstrate certain properties about the integers, the concept of set of integers is needed; yet the creation of integers does not necessitate the creation of sets of integers. Evidently, they must be two different creations. "So here, in order to find out what properties *we* have given to certain objects of our imagination, [we] must first create certain other objects—a very strange situation indeed!" ([1951], p. 314). Faced with the puzzling fact that we must create anew to discover the traits of what we have already created, the force of the suggestion that mathematics is our "free creation" seems undermined.

Thirdly (and again arguing from mathematical practice), Gödel says that "the activity of the mathematician shows very little of the freedom a creator should enjoy." Mathematicians cannot create the validity of theorems at will—"if anything like creation exists at all in mathematics, then what any theorem does is exactly to restrict that freedom" ([1951], p. 314). Any claims to the contrary are at best muddled or confused:

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<sup>46</sup> Gentzen [1936] proved the consistency of Peano Arithmetic using transfinite induction, a new method he developed.

Once it has been made clear exactly *which* objects (including operations, properties and relations) are *in question*, i.e., being talked about, which, all may concede, may well be a matter for choice or decision, the suggestion that there is still room for a decision whether or not those objects have those properties, stand in those relations, etc. cannot be believed to be true. (Boolos [1995], p. 297)

Thus, for example, “Once it is certain that it is 9, 4, 36, multiplication, and equality that are under consideration, how could it possibly be *up to us* whether or not the product of 9 and 4 is 36?” (Boolos [1995], p. 297). If our “creation” could not have turned out any other way, in what sense is it creation at all?

Moschovakis expresses a related idea in his book on set theory: “The main point in favor of the realistic approach to mathematics is the instinctive certainty of most everybody who has ever tried to solve a problem that he is thinking about “real objects”, whether they are sets, numbers, or whatever ...” ([1980], p. 605). The practice of actually doing mathematics appears to correspond best with realism. As Davis and Hersh note:

The typical working mathematician is a [realist] on weekdays and a formalist on Sundays. That is, when he is doing mathematics he is convinced that he is dealing with an objective reality whose properties he is attempting to determine. But when, challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all. ([1981], p. 321)

The philosophy of mathematics is concerned, among other things, with the foundations of mathematics, and the foundations of mathematics is, at least partly, a scientific study of mathematical practice. What mathematicians actually do and actually say is of direct interest. The correlation of mathematical practice to realist sentiments does not necessarily mean that

such sentiments are a reliable indicator of truth, but neither do the complications of giving a philosophical account of realism mean that realism is false. At issue is whether or not mathematics (including its results, methodology, and phenomenology) strengthens the plausibility of realism.

In fact, mathematical results, mathematical practice, and realism all appear to be natural companions:

Realism explains the fact that in actually doing mathematics we frequently experience mathematical statements as meaningful, as having semantic content (not just as meaningless sign configurations); the fact that we take ourselves to have evidence or proof in mathematics, based on an understanding of this meaning (e.g., in set theory), even when we are not working in a rigorous formal system, or in a finitistically acceptable proof theory; the fact that we seem to employ routinely a kind of informal rigor in mathematics, and so on. These are phenomena one would predict if one were a realist about meaning and mathematics ... but not [otherwise]. For [otherwise] these phenomena are anomalous and need to be explained away or reduced. (Tieszen [1994], p. 190)

The disjunctive conclusion, that either the mind is not a finite machine or there exist absolutely undecidable mathematical propositions, is not impervious to criticism. There is a strong case for realism here nonetheless. What is more, supporting Gödel's arguments for realism are intra-mathematical considerations, namely, the incompleteness theorems and faithfulness to mathematical practice.

EPISTEMOLOGY

Any form of realism owes an adequate epistemological explanation. So far, epistemology has been generally neglected, but many of the objections to realism center on it. The primary objection concerns the problem of “accessing” abstract mathematical entities in order to arrive at mathematical knowledge. This “access” problem then runs as follows.<sup>47</sup>

One main way in which mathematical objects differ from their physical counterparts is that they do not figure in spatio-temporal events. They are causally impassive, inactive, and isolated. The mathematical community (among others) is confined to the spatio-temporal realm. This leads to a picture of some “great wall or gulf separating and isolating us on the one side from the mathematical objects on the other side, and the picture leads to the worry, ‘How can we possibly know what is going on over on the other side?’” (Burgess and Rosen [2005], p. 521). How do we “access” abstract mathematical objects? If they are truly isolated, how is mathematical knowledge possible? Either there is no mathematical knowledge, or realism has led us astray.

There are key implicit assumptions to such an epistemological challenge. But before these are considered, Gödel’s own epistemology of mathematics is taken into account.

Underlying Gödel’s epistemology is an analogy between mathematics and natural science, particularly physics. The analogy supports, in Maddy’s phrase, Gödel’s “two-tiered” ([1990], p. 33) epistemology: mathematical intuition,<sup>48</sup> and justification by consequence.

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<sup>47</sup> Paul Benacerraf’s two essays [1965] and [1973] are generally regarded as exemplary versions of this method of epistemological objection.

<sup>48</sup> Gödel’s faculty of mathematical intuition is not intuitionistic intuition (following Brouwer) and is not easily identified with classical rational intuition, as it does not provide an infallible guide to truth. See Parsons [1995] for an in-depth analysis of how Gödel’s faculty of intuition differs from other forms of intuition.

On the first level, Gödel contends for the faculty of mathematical intuition on the basis of actually doing mathematics. Broaching the idea of mathematical intuition is rather non-controversial:

The existence, as a psychological fact, of an intuition covering the axioms of classical mathematics can hardly be doubted.... In fact, this intuition (which may be called the “natural mathematical intuition”) has a high degree of precision, as can be seen from the fact that mathematicians never disagree as to the question of the correctness of a proof, even if they are not familiar with the precise axiomatization of classical mathematics. ([1953/9], version II, pp. 193-194, n. 12)

After demonstrating the presence of intuition in mathematics, he explains that the intuition envisaged issues in our primary source of mathematical knowledge:

... we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them ... ([1964], pp. 483-484)

Mathematical intuition then, works as a guide to the truth of new axioms.

Also, Gödel readily anticipates the possible objection that our intuitions, say, of sets, may be conflicting. The explication he gives is particularly illuminating. Recognizing the depth of his analogy, he reasons that “the set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics” ([1964], p. 484). Though frequently

alleged as a disproof of Platonism, Gödel considers the charge unjust, for “our visual perceptions sometimes contradict out tactile perceptions, for example, in the case of a rod immersed in water, but nobody in his right mind will conclude from that fact that the outer world does not exist” ([1951], p. 321). Criticisms of the faculty of mathematical intuition are not much better than criticisms of sense perception. To the extent that such criticisms lead one away from mathematical realism, they will tend also to lead one away from realism about physical objects. There is no palpable reason then to see the difficulties of mathematical intuition in particular, and a realist epistemology in general, as less tractable than similar hindrances in the physical sciences.

The connection to physics and physical bodies becomes more explicit in Gödel's commentary on Russell's no-class theory. Russell and Whitehead aver that “classes, so far as we introduce them, are merely symbolic or linguistic conveniences, not genuine objects” ([1925], p. 72). But in light of the no-class project's negative results, Gödel maintains,

The assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence.

They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions. ([1944], p. 127)

Gödel later states, parenthetically, that “the question of the objective existence of the objects of mathematical intuition” is to be considered “an exact replica of the question of the objective existence of the outer world” ([1964], p. 484). As put by Kreisel, “The realist assumption of

external mathematical objects is not more *dubious* than that of physical objects” ([1965], p. 186).<sup>49</sup>

As for the second tier, deciding questions of truth and falsity about some new axiom of mathematics is possible in another way—inductively by studying its success:

Success here means fruitfulness in consequences, in particular in “verifiable” consequences, i.e., consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs.... There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems ... that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory. (Gödel [1964], p. 477)

The analogy between mathematics and physical science is present on both levels. Gödel relates mathematical intuition to sense perception and the assumption of mathematical objects to the assumption of physical objects. Also present, though, are purely mathematical concerns. Sense perception “induces us to build up physical theories and to expect that future sense perceptions will agree with them” ([1964], p. 484), and similarly, mathematical intuition induces us to build up mathematical theories. Just as the existence of physical objects is necessary for a satisfactory theory of sense perception, so too are mathematical objects “necessary to obtain a

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<sup>49</sup> Charles Chihara strongly disputes this, what he calls Gödel’s and Kreisel’s “equi-supportive” claim. Chihara argues the even if it is assumed that the case for the existence of mathematical objects is of the same sort as the case for physical objects, neither Gödel nor Kreisel establish that the two cases are equal. Without this, he concludes that there is not as much reasons to believe in one as the other. See Chihara [1973], pp. 60-81 and [1982], pp. 211-227.

satisfactory system of mathematics” ([1944], p. 127). Matters internal to mathematics lead to the mathematics/science analogy.

Of course, there are related ways of exploring the mathematics/science analogy as well. These include the Quinean conception of ontological commitment and the indispensability argument.<sup>50</sup> Putnam expresses its force clearly, declaring that

Quantification over mathematical entities is indispensable for science ... therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question. This type of argument stems, of course, from Quine, who has for years stressed both the indispensability of mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes. ([1971], p. 347)

Abstract mathematical entities are needed to explain our scientific theories of the world—another reason to persist amid epistemologically-pointed objections.

However, behind the original epistemological challenge—the “access” problem—is the assumption and application of a kind of causal theory of knowledge: empirical knowledge is based on perception; perception is a causal relation; this principle must extend to other forms of knowledge, including mathematical. But reasons are not given for why mathematics must be held accountable to some causal theory.

It would be a gross mistake to repudiate the central claims of Mesozoic paleontology or Byzantine historiography on the basis of a theory of justification that had been developed and tested on examples drawn exclusively from, say, particle physics. [Those] who wield a theory of justification developed by reflection on cases of empirical belief as a club against the mathematicians can

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<sup>50</sup> Quine [1953] and [1976].

with considerable plausibility be charged with a similar mistake. (Burgess and Rosen [2005], p. 521)

From the beginning, the possibility of an epistemology particular to mathematics is ruled out. Some version of the causal theory may well be accepted in one sphere or discipline, and yet, “We find differences in methodology between physics and botany—physicists hardly ever take field trips—so the science of mathematics can be expected to have its own techniques as well” (Maddy [1989], p. 1140), including, perhaps, its own epistemology.

Moreover, a causal theory account of empirical knowledge is not itself entirely clear.<sup>51</sup> The burden, then, of describing how it is that one comes to know something (in mathematics or science in general) may in fact be shifted away from the realist and onto his or her critics:

Scientists generally derive their results from ordinary judgments (about their observational instruments and experimental apparatus). But such a derivation provides a justification only if ordinary perceptual judgments are themselves justified. (This is not a “theory” of justification, but just a platitude.) Now has anyone shown that the kind of process by [which] ordinary perceptual judgments are arrived at is a reliable one, tending to lead to *true* judgments? Have ordinary perceptual judgments been justified? Not by the corps of scientists, who never consider such questions. Only a small cadre of specialists, with both feet in philosophy, ever bother to consider the epistemic status of perceptual judgments, and even they do not manage to agree among themselves as to what the justification for them is supposed to be. (Burgess and Rosen [2005], p. 523)

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<sup>51</sup> As Bonjour observes: “Though the theses in question [i.e., causal accounts of empirical knowledge] are often treated as though they were obvious and unproblematic, the precise nature of the evidence or other basis for accepting them is often very uncertain” ([1998], p. 154).

Thus there are multiple ways to answer the original question. Though requiring an epistemological account is surely warranted, and providing a compelling one certainly requires much critical thought, the realist is not left defenseless from the outset. From Gödel's perspective, intuition and fruitfulness provide the necessary sources of mathematical knowledge. So it is reasonable to conclude that simply raising the epistemological pitfalls of realism is not sufficient grounds for realism's dismissal. After all, a causal theory of empirical knowledge is not without its own problems as well.

## CONCLUSION

In the foundations of mathematics, the influence of Gödel's work is extraordinary. In a statement delivered at a commemoration service for Gödel held at the Institute for Advanced Study in March of 1978 Simon Kochen declared,

His results can be counted on the fingers of one hand.... Professor Kleene was on my generals committee ... and his first question for me was: "Give five theorems of Gödel." Just five theorems. But what monumental and incredible theorems those are. Each ... is the beginning of a whole branch of modern mathematical logic. (Dawson [1998], p. 4)

The influence of realism on Gödel's work calls for documentation as well. "We do have the striking example of Gödel who possesses firmly held philosophical views which played an essential role in making his fundamental new scientific discoveries, and who is well aware of the importance of his philosophical views for his scientific work" (Wang [1974], p. 8). But the path of influence does not move just from Gödel's theorems to modern mathematical logic or from

realism to the incompleteness theorems. Another line moves from mathematics to philosophy. Though certain hurdles appear along the trail, such as giving a precise elucidation of the first alternative in the disjunctive conclusion of Gödel's Gibbs Lecture, the fact that it points towards realism is clear. Our guide, the two lines of reasoning in Gödel's thought (intra-mathematical and extra-mathematical), leads toward the same conclusion. In other words, judging by Gödel's work in mathematics, in particular the incompleteness theorems, it seems very likely that mathematics and some form of realism are best understood hand-in-hand.

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